Shearlets as Multi-scale Radon Transform

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***Abstract*—We show that the 2D-shearlet transform is the composition of the affine Radon transform, a 1D-wavelet transform and a 1D-convolution.**

I. INTRODUCTION

Many effective algorithms in signal analysis, image processing and computer vision are based on efficient multi-scale representations. It is well known that wavelets provide an almost optimal framework for 1D signals, whereas for multi-dimensional signals a huge class of representations has been introduced like directional wavelets [1], ridgelets [2], curvelets [3], wavelets with composite dilations [4], contourlets [5], shearlets [6], reproducing groups of the symplectic group [7], Gabor ridge functions [7] and mocklets [8] – to name a few.

Among them, shearlets emerge because of their ability to efficiently capture anisotropic features, to provide an optimal sparse representation, to detect singularities and to be stable against noise. The effectiveness of shearlets is supported by the well-established mathematical theory of square-integrable representations and it is tested in many applications in image processing, where many efficient algorithms based on shearltes have been designed (see [9], [10] and also the website http://www.shearlab.org/ for further details and references).

For these reasons, it is natural to observe that shearlets for 2D-signals behave as wavelets for 1D-signal, so that one could try to understand if this strong connection is a consequence of some general mathematical principle.

The aim of this paper is to show that the Radon transform is the link between the shearlet transform and the 1D-wavelet trasform since it intertwines the shearlet representation with a suitable tensor product of two wavelet representations. This relationship is part of a series of issues that has already been addressed. Indeed it is known that ridgelets are constructed via wavelet analysis in the Radon domain [11], Gabor frames are defined as the directionally-sensitive Radon transforms [7], discrete shearlet frames are used to invert the Radon transform [12] and the Radon transform is at the root of the proof that shearlets are able to detect the wavefront set of a 2D signal [13]. Our contribution is to clarify this relation from the point of view of non-commutative harmonic analysis.

The paper is organised as it follows. In Section 2 we briefly recall the wavelet, shearlet and Radon transforms. In Section 3 we state the main results, and in Section 4 we show further results for d-dimensional signals. Finally, Section 5 contains a sketch of the proofs and Section 6 some concluding remarks.

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II. WAVELET, SHEARLET AND RADON TRANSFORMS: AN OVERVIEW

We briefly introduce the notation. We set $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, regarded as multiplicative group. The scalar product and the norm of $\mathbb{R}^d$ are denoted by $\vec{n} \cdot \vec{n}'$ and $|\vec{n}|$, respectively. We denote by $L^p(\mathbb{R}^d)$ the Banach space of functions $f : \mathbb{R}^d \to \mathbb{C}$, which are p-integrable with respect to the Lebesgue measure $d\vec{x}$ and, if $p = 2$, the corresponding scalar product and norm are $\langle f, g\rangle$ and $\|f\|$. The Fourier trasform is denoted by $\mathcal{F}$ both on $L^2(\mathbb{R}^d)$ and on $L^1(\mathbb{R}^d)$, where it is defined by

$$\mathcal{F}f(\vec{\xi}) = \int_{\mathbb{R}^d} f(x, y)\mathrm{e}^{-2\pi i\,\vec{\xi}\cdot\vec{x}}\mathrm{d}\vec{x}, \qquad f \in L^1(\mathbb{R}^d).$$

If G is a locally compact group, $L^2(G)$ is the Hilbert space of square-integrable functions with respect to a left Haar measure.

A. Wavelets

The wavelet group is $\mathbb{W} = \mathbb{R} \rtimes \mathbb{R}^*$ with group law

$$(b, a)(b', a') = (b + ab', aa').$$

The wavelet representation W acts on $L^2(\mathbb{R})$ as

$$W_{b,a}\psi(x) = |a|^{-1/2}\psi(\frac{x-b}{a}), \qquad x \in \mathbb{R},$$

and the wavelet transform $\mathcal{W}_\psi : L^2(\mathbb{R}) \to L^2(\mathbb{W})$

$$\mathcal{W}_\psi f(b, a) = \langle f, W_{b,a}\psi\rangle, \quad (b, a) \in \mathbb{W}$$

is a (non-zero) multiple of isometry provided that

$$0 < \int_{\mathbb{R}} \frac{|\mathcal{F}\psi(\xi)|^2}{|\xi|}\,\mathrm{d}\xi < +\infty.$$

In this case, ψ is called an admissible wavelet [14], [15].

B. Shearlets

Given $\gamma \in \mathbb{R}$, the shearlet group is $\mathbb{S}^\gamma = \mathbb{R}^2 \rtimes (\mathbb{R} \rtimes \mathbb{R}^*)$ with group law

$$(\vec{b}, s, a)(\vec{b}', s', a') = (\vec{b} + N_sA_a\vec{b}', s + |a|^{1-\gamma}s', aa')$$

where

$$A_a = a\begin{bmatrix}1 & 0\\ 0 & |a|^{\gamma-1}\end{bmatrix} \quad N_s = \begin{bmatrix}1 & -s\\ 0 & 1\end{bmatrix}$$

and the vectors are understood as column vectors. Parabolic shearlets, which were introduced in [16], correspond to the choice $\gamma = 1/2$. The group $\mathbb{S}^\gamma$ acts on $L^2(\mathbb{R}^2)$ as

$$S^\gamma_{\vec{b},s,a}f\,(\vec{x}) = |a|^{-\frac{1+\gamma}{2}} f(A_a^{-1}N_s^{-1}(\vec{x} - \vec{b})), \qquad \vec{x} \in \mathbb{R}^2,$$

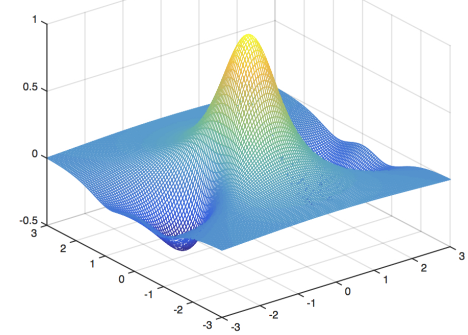

Fig. 1. A *classical* mother shearlet ψ in the space domain.

and the shearlet transform $\mathcal{S}^\gamma_\psi : L^2(\mathbb{R}^2) \to L^2(\mathbb{S}^\gamma)$

$$\mathcal{S}^\gamma_\psi f(\vec{b}, s, a) = \langle f, S^\gamma_{\vec{b},s,a}\psi\rangle, \qquad (\vec{b}, s, a) \in \mathbb{S}^\gamma$$

is a (non-zero) multiple of an isometry provided that

$$0 < \int_{\mathbb{R}^2} \frac{|\mathcal{F}\psi(\xi_1,\xi_2)|^2}{|\xi_1|^2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 < +\infty. \tag{1}$$

Classical mother shearlets [16] are of the form

$$\mathcal{F}\psi(\xi_1,\xi_2) = \mathcal{F}\psi_1(\xi_1)\mathcal{F}\psi_2\left(\frac{\xi_2}{\xi_1}\right)$$

where ψ_1 is an admissible wavelet and $\mathcal{F}\psi_2$ is a bump function in the Fourier domain [17]. An example of shearlet ψ is depicted in Fig. 1. In [18] a different choice for the mother wavelets has been proposed to have compactly supported shearlets in the space domain.

C. Radon transforms

The Radon transform [19] is usually defined by parametrising the lines by pairs $(\theta, q) \in [-\pi, \pi) \times \mathbb{R}$, that is

$$\Gamma_{\theta,q} = \{(x, y) \in \mathbb{R}^2 \mid x\cos\theta + y\sin\theta = q\}.$$

To stress the dipendency on the polar angle θ we write

$$\mathcal{R}^{\mathrm{pol}} f(\theta, q) = \int_{x\cos\theta + y\sin\theta = q} f(x, y)\,\mathrm{d}\ell(x, y),$$

where $\mathrm{d}\ell$ is the measure on the line $\Gamma_{\theta,q}$. We label the normal vector to a line by affine coordinates, see Fig. 2, writing

$$\Gamma_{v,t} = \{(x, y) \in \mathbb{R}^2 \mid x + vy = t\}$$

where the correspondence is $v = \tan\theta$ and $t = q/\cos\theta$. With this parametrisation, the vertical lines, which correspond to the choice $\theta = \pm\pi/2$, can not be represented, but they constitute a negligible set with respect to the natural measure on the affine projective space $\mathbb{P}^1 \times \mathbb{R} = \{\Gamma \mid \Gamma \text{ line of } \mathbb{R}^2\}$.

The (affine) Radon transform of any $f \in L^1(\mathbb{R}^2)$ is the function $\mathcal{R}f : \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\mathcal{R}f(v, t) = \int_{\mathbb{R}} f(t - vy, y)\,\mathrm{d}y, \qquad \text{a.e. } (v, t) \in \mathbb{R}^2.$$

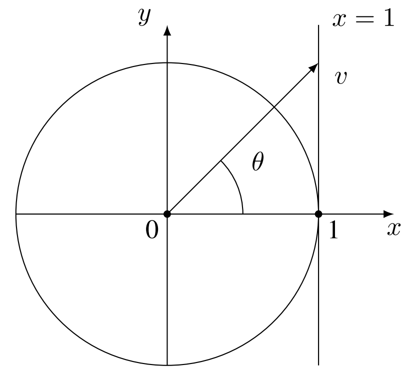

Fig. 2. The affine coordinates

and is related to the (polar) Radon transform by

$$\mathcal{R}f(v, t) = \frac{1}{\sqrt{1+v^2}} \mathcal{R}^{\mathrm{pol}} f(\arctan v, \frac{t}{\sqrt{1+v^2}}). \tag{2}$$

The central Fourier slice theorem [19] shows that the Radon transform is strongly related to the Fourier trasform since for any $f \in L^1(\mathbb{R}^2)$ it holds that

$$(I \otimes \mathcal{F})\mathcal{R}f(v, \omega) = \mathcal{F}f(\omega, \omega v) \qquad (\omega, v) \in \mathbb{R}^2. \tag{3}$$

As in the case of the Fourier transform, it is possible to extend $\mathcal{R}$ to $L^2(\mathbb{R}^2)$ as a unitary map. However, this raises some technical issues. First, consider the subspace

$$\mathcal{D} = \{g \in L^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} |\omega||(I \otimes \mathcal{F})g(v, \omega)|^2 \mathrm{d}v\mathrm{d}\omega < +\infty\},$$

which is a dense subset of $L^2(\mathbb{R}^2)$, and then define the self-adjoint unbounded operator $\mathcal{J} : \mathcal{D} \to L^2(\mathbb{R}^2)$ by

$$(I \otimes \mathcal{F})\mathcal{J}F(v, \omega) = |\omega|^{\frac{1}{2}}(I \otimes \mathcal{F})F(v, \omega), \qquad \text{a.e. } (v, \omega) \in \mathbb{R}^2,$$

which is a Fourier multiplier with respect to the second variable. Then, it is not hard to show that for all f in the dense subspace of $L^2(\mathbb{R}^2)$

$$\mathcal{A} = \{f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} \frac{|\mathcal{F}f(\xi_1,\xi_2)|^2}{|\xi_1|} \mathrm{d}\xi_1\mathrm{d}\xi_2 < +\infty\},$$

the Radon transform $\mathcal{R}f$ belongs to $\mathcal{D}$ and the map

$$f \longmapsto \mathcal{J}\mathcal{R}f$$

from $\mathcal{A}$ to $L^2(\mathbb{R}^2)$ extends to a unitary map, denoted by $\mathcal{Q}$, from $L^2(\mathbb{R}^2)$ onto itself. We need the following generalisation of the Fourier slice theorem.

Corollary 1. *For all* $f \in L^2(\mathbb{R}^2)$

$$(I \otimes \mathcal{F})\mathcal{Q}f(v, \omega) = |\omega|^{\frac{1}{2}}\mathcal{F}f(\omega, \omega v), \qquad \textit{a.e. } (\omega, v) \in \mathbb{R}^2. \tag{4}$$

If $f \in \mathcal{A}$, (4) is an easy consequence of (3) and the definition of $\mathcal{J}$, and this is known (see [12], Section 3.2 and [20]). For arbitrary $f \in L^2(\mathbb{R}^2)$ the proof is not trivial because $\mathcal{Q}$ cannot be written as $\mathcal{J}\mathcal{R}$, and is based on the fact that $\mathcal{J}$ is a Fourier multiplier.

III. MAIN RESULTS

The next result shows that $\mathcal{Q}$ intertwines the shearlet representation S^γ with the tensor product of two wavelet representations W.

Theorem 2. *For all $(\vec{b}, s, a) \in \mathbb{S}^\gamma$ and $f \in L^2(\mathbb{R}^2)$*

$$\mathcal{Q}\, S^\gamma_{\vec{b},s,a} f = \big(W_{s,|a|^{1-\gamma}} \otimes \mathrm{I}\big)\, W_{(1,\mathbf{v})\cdot\vec{b},a}\, \mathcal{Q} f \tag{5}$$

In the above theorem the letter $\mathbf{v}$ in the right hand side is a dummy variable, *i.e.*, the action on a function $F \in L^2(\mathbb{R}^2)$ is

$$W_{(1,\mathbf{v})\cdot\vec{b},a} F(v,t) = |a|^{-\frac{1}{2}} F\left(v, \frac{t-(1,v)\cdot\vec{b}}{a}\right).$$

Eq. (5) suggests that a natural choice for the admissible vector ψ is of the form

$$\mathcal{Q}\psi = \phi_2 \otimes \phi_1$$

where $\phi_1, \phi_2 \in L^2(\mathbb{R})$. As a consequence of (4),

$$\mathcal{F}\psi(\xi_1,\xi_2) = \mathcal{F}\psi_1(\xi_1)\mathcal{F}\psi_2(\frac{\xi_2}{\xi_1})$$

where

$$\mathcal{F}\phi_1(\omega) = |\omega|^{\frac{1}{2}}\mathcal{F}\psi_1(\omega) \qquad \phi_2(v) = \mathcal{F}\psi_2(v).$$

The admissibility condition (1) is equivalent to the following requirements on ψ_1 and ψ_2

$$0 < \int_{\mathbb{R}} \frac{|\mathcal{F}\psi_1(\omega)|^2}{|\omega|}\mathrm{d}\omega < +\infty$$
$$\int_{\mathbb{R}} |\omega||\mathcal{F}\psi_1(\omega)|^2 \mathrm{d}\omega < +\infty$$
$$0 < \int_{\mathbb{R}} |\mathcal{F}\psi_2(v)|^2\, \mathrm{d}v < +\infty.$$

This means that the function ψ_1 is a 1D-wavelet in the (fractional) Sobolev space $H^{\frac{1}{2}}(\mathbb{R})$ and the function $\mathcal{F}\psi_2$ is a square-integrable function with respect to the variable $v = \xi_2/\xi_1$ in the Fourier domain. Furthermore, it is possible to prove that ϕ_1 is an admissible wavelet, too, and that

$$\mathcal{S}^\gamma_\psi(f)(x,y,s,a) = |a|^{\frac{\gamma-1}{2}} \int_{\mathbb{R}} \mathcal{W}_{\phi_1}\big(\mathcal{Q}f(v,\bullet)\big)(x+vy,a) \times \overline{\times\, \phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)}\, \mathrm{d}v, \tag{6}$$

where $\mathcal{W}_{\phi_1}\big(\mathcal{Q}f(v,\bullet)\big)$ means that the wavelet transform is computed with respect to the second variable. If ϕ_2 is a bump function, the behaviour of the integral over v depends on the value of γ. Indeed, if a goes to zero and $\gamma < 1$, $\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)$ is an approximation of the identity, wheres if $\gamma > 1$, it looks as a scale dependent smoothing. In signal analysis usually $\gamma = 1/2$.

Eq. (6) is not easy to implement in applications since the definition of $\mathcal{Q}$ involves both a limit and the pseudo differential operator $\mathcal{J}$. However, if $\phi_2 \otimes \phi_1$ is in the domain of $\mathcal{J}$, we can set $\phi_2 \otimes \chi_1 = \mathcal{J}(\phi_2 \otimes \phi_1)$, *i.e.*

$$\mathcal{F}\chi_1(\omega) = |\omega|\mathcal{F}\psi_1(\omega), \tag{7}$$

so that $\chi_1 = \frac{1}{2\pi}\,\mathrm{H}\,\psi_1'$, where $'$ is the weak derivative and H is the Hilbert transform. With this choice, χ_1 is an admissible wavelet, too, and it holds that

$$\mathcal{S}^\gamma_\psi f(x,y,s,a) = |a|^{\frac{\gamma-2}{2}} \int_{\mathbb{R}} \mathcal{W}_{\chi_1}\big(\mathcal{R}f(v,\bullet)\big)(x+vy,a) \times \overline{\times\, \phi_2\Big(\frac{v-s}{|a|^{1-\gamma}}\Big)}\, \mathrm{d}v \tag{8}$$

for all $f \in \mathcal{A}$. The above formula can be written in terms of the polar Radon transform using (2) and can actually be extended to $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ (see [21]).

Eq. (8) shows that for any signal f in $\mathcal{A}$ the shearlet coefficients can be computed by means of three *classical* trasforms.

a) Compute the Radon transform $\mathcal{R}f(v,t)$.
b) Apply the wavelet transform with respect to the variable t

$$G(v,\vec{b},a) = \mathcal{W}_{\chi_1}\big(\mathcal{R}f(v,\cdot)\big)(\vec{b},a). \tag{9}$$

where χ_1 is given by (7).
c) Convolve the result with the scale-dependent filter

$$\Phi_a(v) = \overline{\phi_2\left(-\frac{v}{|a|^{1-\gamma}}\right)},$$

where the convolution is computed with respect to the variable v,

$$\mathcal{S}^\gamma_\psi f(x,y,s,a) = \left(G(\bullet, x+\bullet\, y, a) * \Phi_a\right)(s). \tag{10}$$

Finally, since $\mathbb{S}^\gamma$ is a square-integrable representation, there is a reconstruction formula. Indeed

$$f = \int_{\mathbb{S}^\gamma} \mathcal{S}^\gamma_\psi f(x,y,s,a)\; S^\gamma_{x,y,s,a}\psi\; \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}s\mathrm{d}a}{|a|^3},$$

where the integral converges in the weak sense. Note that $\mathcal{S}^\gamma_\psi f(x,y,s,a)$ depends on f only through its Radon transform $\mathcal{R}f$ (see (9) and (10)). The above equation allows to reconstruct an unknown signal f from its Radon transform by computing the shearlet coefficients by means of (8).

IV. FURTHER EXTENSIONS

The construction can be extended to the generalised shearlet groups introduced in [22]. This class of groups consists of semi-direct products $G = \mathbb{R}^d \rtimes H$. The homogenous group H is a closed subgroup of $\mathrm{GL}(d,\mathbb{R})$ of the form

$$H = \{a \begin{bmatrix} 1 & -{}^t\vec{s}\,\Lambda(a) \\ 0 & B(\vec{s})\,\Lambda(a)\end{bmatrix} = h_{\vec{s},a} \mid a \in \mathbb{R}^*, \vec{s} \in \mathbb{R}^{d-1}\},$$

where $\Lambda(a)$ is a diagonal matrix of size $d-1$

$$\begin{bmatrix} |a|^{\lambda_1} & 0 & \dots & 0 \\ 0 & |a|^{\lambda_2} & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & \dots & \dots & |a|^{\lambda_{d-1}} \end{bmatrix}$$

and $B(\vec{s})$ is a unipotent upper triangular matrix of size $d-1$

$$B(\vec{s}) = \begin{bmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ 0 & \dots & \ddots & * \\ 0 & \dots & \dots & 1 \end{bmatrix}.$$

For example, the shearlet group for d-dimensional signals introduced in [23] corresponds to the choices $B(\vec{s}) = \mathrm{I}_{d-1}$ and $\lambda_1 = \ldots = \lambda_{d-1} = 1/d - 1$. If $d = 2$ we get the shearlet group introduced in Section II.

Remark 1. The group H is the semi-direct product of the normal subgroup

$$S = \{ \begin{bmatrix} 1 & -{}^t\vec{s} \\ 0 & B(\vec{s}) \end{bmatrix} \mid \vec{s} \in \mathbb{R}^{d-1} \}$$

and the abelian subgroup

$$D = \{ a \begin{bmatrix} 1 & 0 \\ 0 & \Lambda(a) \end{bmatrix} \mid a \in \mathbb{R}^* \}.$$

Clearly, D is isomorphic, as a Lie group, to $\mathbb{R}^*$. For classical shearlets, S is isomorphic as a Lie group to the additive abelian group $\mathbb{R}^{d-1}$. In the general setting, S is diffeomorphic to $\mathbb{R}^{d-1}$ only as a manifold. In order to stress the symmetry between the general case and the classical shearlet group, we identify H and $\mathbb{R}^{d-1} \times \mathbb{R}^*$ as a manifold and we denote the element $h_{\vec{s},a}$ with the pair $(\vec{s}, a)$. We observe that $(\vec{s}, a) = (\vec{s}, 1)(\vec{0}, a)$ since $h_{\vec{s},a} = h_{\vec{s},1} h_{0,a}$, however in general

$$(\vec{s}, 1)(\vec{s}\,', 1) \neq (\vec{s} + \vec{s}\,', 1).$$

The map $\vec{s} \mapsto B(\vec{s})$ has to satisfy some conditions to ensure that S is a subgroup of $\mathrm{GL}(d, \mathbb{R})$. Furthermore, suitable compatibility conditions between the matrices B and the matrices Λ must be satisfied. A complete characterization of the maps $B(\cdot)$ and $\Lambda(\cdot)$ is given in [22] under the assumption that S is abelian.

The group G acts on $L^2(\mathbb{R}^d)$ as

$$\pi_{\vec{b},\vec{s},a} f(\vec{x}) = |a|^{-\frac{d+\lambda_1+\ldots+\lambda_{d-1}}{2}} f(h_{\vec{s},a}^{-1}(\vec{x} - \vec{b})).$$

Eq. (5) is replaced by

$$\mathcal{Q}\, \pi_{\vec{b},\vec{s},a} f = (V_{\vec{s},a} \otimes \mathrm{I})\, W_{(1,\mathbf{v})\cdot\vec{b},a}\, \mathcal{Q} f,$$

where V is the representation of H on $L^2(\mathbb{R}^{d-1})$ defined by

$$V_{\vec{s},a} F(\vec{v}) = |a|^{\frac{\lambda_1+\ldots+\lambda_{d-1}}{2}} F\left(\Lambda(a) \left({}^t B(\vec{s})\vec{v} - \vec{s}\right)\right).$$

V. Sketch of the proof

We give an idea of the proof of Theorem 2. By continuity it is enough to show (5) assuming that $f \in \mathcal{A}$, so that $\mathcal{Q} = \mathcal{J}\mathcal{R}$.

The following covariance properties of the Radon transform are an easy consequence of suitable changes of variables. For any translation $\vec{b} \in \mathbb{R}^2$ it holds

$$\mathcal{R}\left(f(\cdot - \vec{b})\right)(v, t) = \mathcal{R}[f](v, t - (1, v) \cdot \vec{b});$$

for any invertible diagonal matrix

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \qquad a_1, a_2 \in \mathbb{R}^* \tag{11}$$

it holds

$$\mathcal{R}\left(f(A^{-1}\cdot)\right)(v, t) = |a_2|\, \mathcal{R} f(\frac{a_2}{a_1} v, a_1 t),$$

and for any shearing matrix

$$N = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}, \qquad s \in \mathbb{R}$$

it holds

$$\mathcal{R}\left(f(N^{-1}\cdot)\right)(v, t) = \mathcal{R}[f](v - s, t).$$

Moreover the operator $\mathcal{J}$ clearly commutes with translations and shearings, whereas, if A is as in (11),

$$\mathcal{J}\left(F(A^{-1}\cdot)\right)(v, \omega) = |a_2|^{\frac{1}{2}} \mathcal{J} F(a_1^{-1} v, a_2 \omega).$$

The final step is to observe that, if $f \in \mathcal{A}$, then

$$\mathcal{Q}\, S^{\gamma}_{\vec{b},s,a} f = \mathcal{J}\mathcal{R} S^{\gamma}_{\vec{b},0,1} S^{\gamma}_{0,s,1} S^{\gamma}_{0,0,a} f$$

and apply three times the above covariance relations.

VI. Conclusions

In this paper we show that the shearlet transform of a 2D-signal can be realised by applying first the affine Radon transform, then by computing a 1D-wavelet transform and, finally, by performing a 1D-convolution. This result has been extended to higher dimensional shearlet transforms both in the classical case and for generalized shearlet dilation groups [21]. This relation can give rise to a new algorithm to compute the shearlet coefficients based on the efficient codes available both for the Radon transform and for the wavelet transforms. Furthermore, it opens the possibility to recover a signal from its Radon transform by using the shearlet inversion formula. The application of these findings to image processing tasks is currently under investigation.

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